Allenamento olimpico

Note Title — OSIMO, 06/02/2017

Combinatoria

① Codici del bancomat

bancomat PIN _ _ _ _ _ 5 cifre

Quanti codici? 10^5

□ □ □ □ □

10 possibilità, 10 poss.

Parole in un dato alfabeto.

Quante parole di K lettere posso scrivere in un alfabeto di n caratteri? $\boxed{n^k}$

Gara di corsa.

Ho 4 persone che fanno una gara di corsa Alberto, Bruno, Carlo e Davide.

Quante possibili classifiche diverse ci possono essere?

1° posto ho 4 possibilità (A, B, C, D)

2° posto 3 poss. (B, C, D)

3° posto 2 poss.

4° posto 1 poss.

$4 \cdot 3 \cdot 2 \cdot 1 = 4!$

n concorrenti $\longrightarrow$ $n!$ classifiche

TARGHE:

$$\underset{22}{\square}\;\underset{22}{\square}\;\overset{10}{\_}\;\underset{10}{\_}\;\underset{10}{\_}\;\overset{22}{\square}\;\overset{22}{\square}$$

22 LETTERE

10 CIFRE

$$22^2 \cdot 10^3 \cdot 22^2 = 22^4 \cdot 10^3$$

A, B, C, D, E, F, G $\qquad 7$

$$\underset{7}{\square}\;\underset{6}{\square} \cdot 5 \cdot 4 \quad \dots \quad = 7!$$

A, B, C, D, E, F_1, F_2 $\qquad 7!$

$$\begin{bmatrix} A & B & F_1 & C & D & F_2 & E \\ A & B & F_2 & C & D & F_1 & E \end{bmatrix} \Longrightarrow \frac{7!}{2}$$

$$\begin{bmatrix} \quad \end{bmatrix}$$

$A, B, C, D, \;, F_1, F_2, F_3$

$$\begin{bmatrix} A & B & C & D & F_1 & F_2 & F_3 \\ " & " & " & & F_2 & F_3 & F_1 \\ & & & & F_3 & F_1 & F_2 \\ & & & & \vdots & & \end{bmatrix} \quad 3! \qquad \boxed{\frac{7!}{3!}}$$

$A, B, C_1, C_2, F_1, F_2, F_3$

$$\frac{7!}{\boxed{3!} \cdot 2!}$$

10 GIOCATORI $\longrightarrow$ $10!$

1°	2°	3°
$\square$	$\square$	$\square$
$\uparrow$	$\uparrow$	$\uparrow$
10	9	8

$$\boxed{10 \cdot 9 \cdot 8} = \frac{10!}{7!}$$

MATEMATICA

$$\frac{10!}{3!\,2!\,2!}$$

MATEMATICA

tutte le A vicine

A A A C E I M M T T

@

simboli 8

$$\frac{8!}{\underset{M}{2!}\,\underset{T}{2!}}$$

tutte le vocali vicine

VVVVV C M M T T $\longrightarrow$ $\underbrace{\boxed{\frac{6!}{2!\,2!}}}_{\text{ordini dei 6 simboli}} \cdot \underbrace{\frac{5!}{3!}}_{\text{ordini delle vocali}}$

- tutte le cons. in ordine alfabetico

$$\underbrace{\frac{10!}{3!\,2!\,2!}}_{\text{an. MAT.}} : \underbrace{\frac{5!}{2!\,2!}}_{\text{an. conson.}} = \frac{10!}{3!\,5!}$$

KAKEKAKIKA $\qquad \dfrac{10!}{5!\,3!}$

ES 9

$6\left\{ \begin{array}{l} 1\ 2\ 3 \leftarrow \\ \vdots \end{array}\right.$ $\qquad \dfrac{90\cdot 89\cdot 88}{6}$

$90 \longrightarrow 3$

$n \longrightarrow k$

10 PENNE DISTINTE $\qquad 1\ 2\ 3\ \ldots\ 10$

NE SCELGO 5

$\begin{bmatrix} 5 & 7 & 8 & 9 & 1 \\ 1 & 5 & 7 & 9 & 8 \\ & & \vdots & & \end{bmatrix} \longrightarrow 5!$ $\qquad \dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5!} = \dfrac{10!}{5!\cdot 5!}$

n OGGETTI DISTINTI $\qquad 1, 2, 3, 4, 5, \ldots, n$

$\begin{bmatrix} \overbrace{1, 2, 7, 4, \ldots, 9, 1}^{k} \\ \vdots \end{bmatrix} \quad k!$

$\square \ \square \ \square \ \ldots \ \square$

$1° \ \ 2° \ \ 3° \ \ldots \ k°$

$$\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

ES $\binom{n}{k} = \binom{n}{n-k}$

$n = 7$

$k = 0 \qquad n-k = 7 \qquad \dfrac{7!}{0!\,(7-0)!}$

$0! = 1$

$\binom{n}{k}$

$$
\begin{array}{l|ccccccccccccc}
0 & & & & & & & 1 & & & & & & \\
1 & & & & & & 1 & & 1 & & & & & \\
2 & & & & & 1 & & 2 & & 1 & & & & \\
3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

$\binom{4}{1} = 4$

$$
\begin{array}{ccccccccccc}
\binom{n+1}{2} = & \binom{1}{1} & + & \binom{2}{1} & + & \binom{3}{1} & + & \binom{4}{1} & + \dots & \binom{n}{1} \\
 & \shortparallel & & \shortparallel & & \shortparallel & & & & \shortparallel \\
 & 1 & & 2 & & 3 & & & & n
\end{array}
$$

$$\binom{n+1}{2} = 1+2+3+4 \dots + n$$

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} \dots + \binom{7}{7} = 2^n$$

$$\boxed{\binom{7}{0} - \binom{7}{1} + \binom{7}{2} - \dots - \binom{7}{7} = ?}$$

→ IN QUANTI MODI POSSO SCEGLIERE UN SOTT DA UN INSIEME D n ELEMENTI?

$$\{1, 2, 3, 4, \dots, n\}$$

$$\boxed{\frac{SI}{NO}}\ \boxed{\frac{SI}{NO}} \ \dots \ \boxed{\frac{SI}{NO}} \quad \rightarrow \underbrace{2 \cdot 2 \cdot 2 \cdot \dots \cdot 2}_{n} = 2^n$$

$\binom{n}{k}$

$$
\begin{array}{l|ccccccccccccc}
0 & & & & & & & 1 & & & & & & \\
1 & & & & & & 1 & & 1 & & & & & \\
2 & & & & & 1 & & 2 & & 1 & & & & \\
3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 & 1 & & 6 & & 15 & & \boxed{20} & & 15 & & 6 & & 1
\end{array}
$$

$$\binom{n+1}{2+1} = \underbrace{\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \dots \binom{n}{2}}$$

$$+\cancel{\binom{7}{0}} - \cancel{\binom{7}{1}} + \cancel{\binom{7}{2}} - \cancel{\binom{7}{3}} + \cancel{\binom{7}{4}} - \cancel{\binom{7}{5}} + \cancel{\binom{7}{6}} - \cancel{\binom{7}{7}} = 0$$

Geometria

7/2/2017

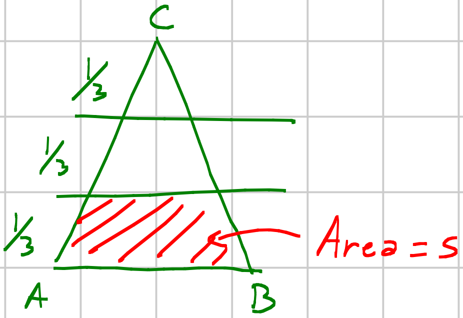
www2.unipr.it/~saral674 (Divulgazione)

poisson.phc.unipi.it/~gori de/osimo.html

Problemi

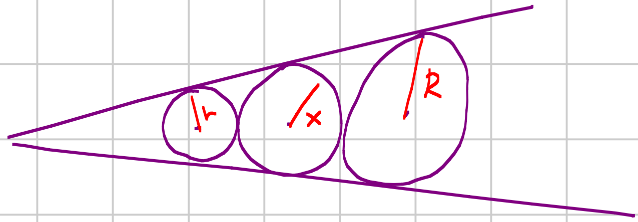
1. Le rette nel disegno sono parallele ad AB.

$A(\widehat{ABC}) = ?$



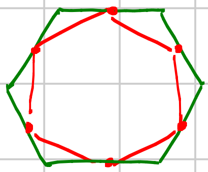
2. Tutto tangente.

$x = ?$



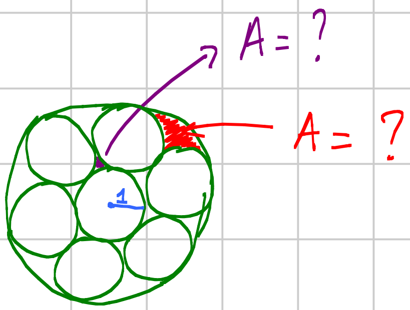
3. ⬡ esagono regolare

• punti medi lati



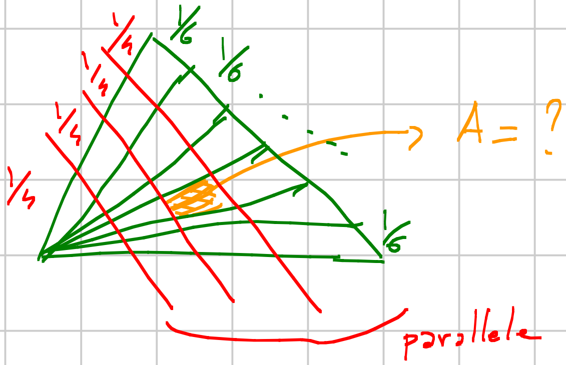
$\frac{A}{A} = ?$

4.

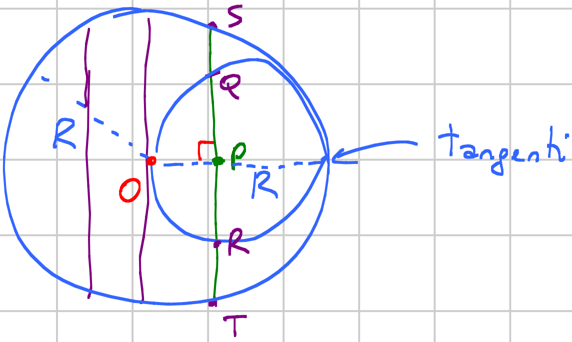


5.

Area △ = 1



6.

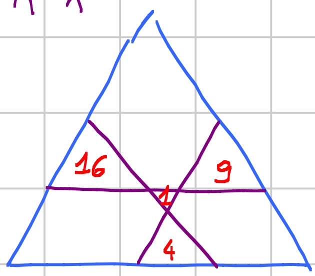


$\overline{QR} = \frac{1}{2}\overline{ST}$

$\overline{OP} = ?$

$$\frac{R+x}{2} = R - x$$

7.

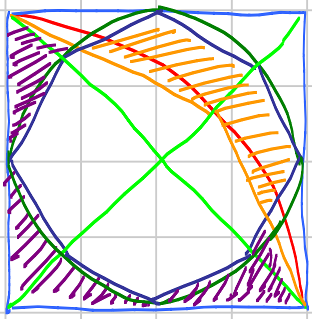


Aree

equilatero

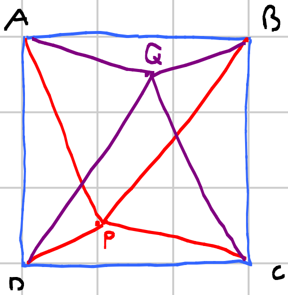
$A(\triangle) = ?$

8.



$A - A = ?$

9.



$\overline{AQ}^2 + \overline{BQ}^2 + \overline{CQ}^2 + \overline{DQ}^2 = 594$

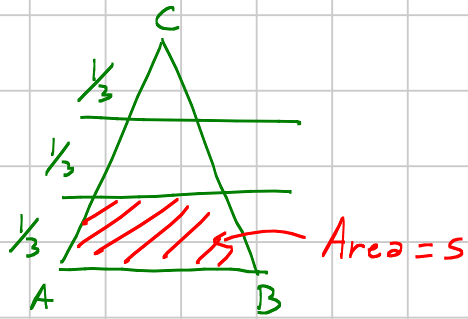
$\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = 499$

PERIMETRO(ABCD) = 60

DOMANDA: QUANTO VALE AL MASSIMO $\overline{PQ}$?

1. Le rette nel disegno sono parallele ad AB.

$A(\widehat{ABC}) = ?$



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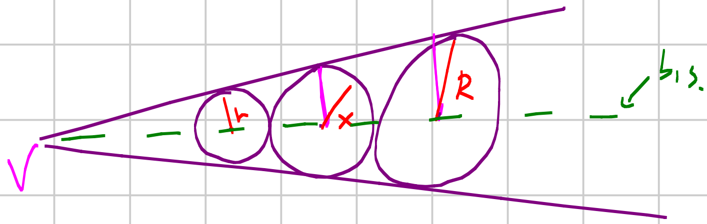
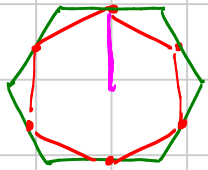


$$X = \sqrt{R\,r}$$

3. ⬡ esagono regolare

• punti medi lati



$$\frac{A}{A} = ?$$

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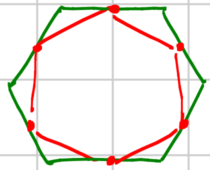

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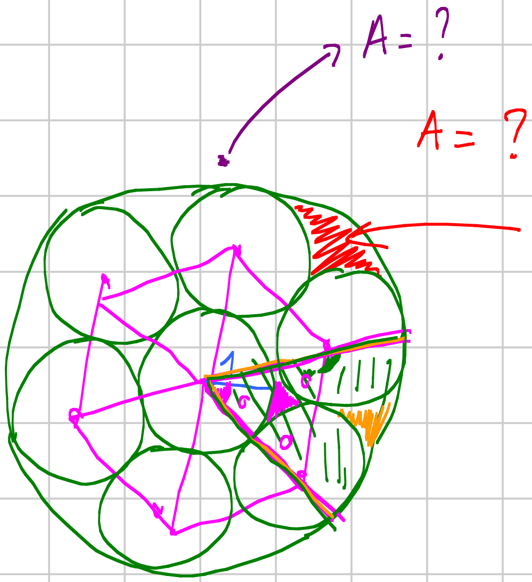

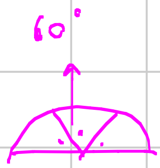

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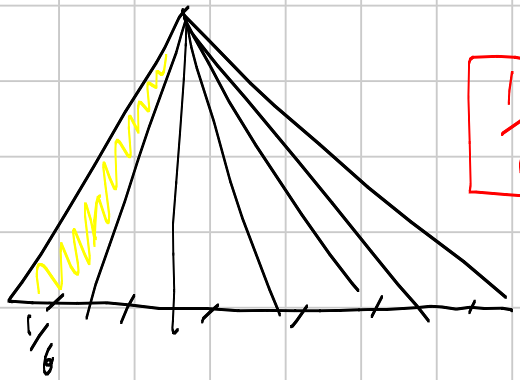
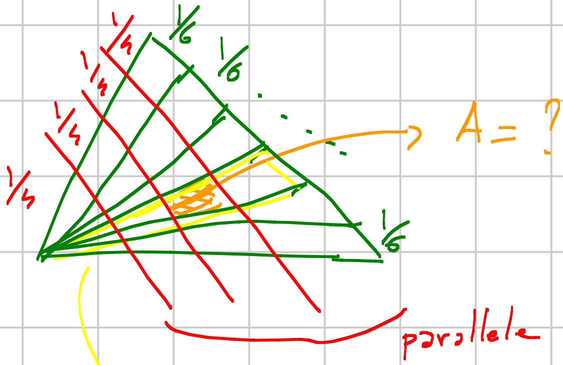
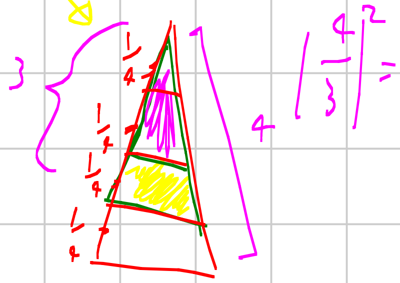
Area △ = 1



$\boxed{\frac{1}{6}}$





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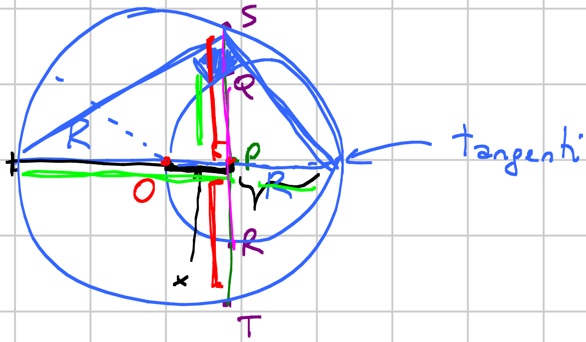


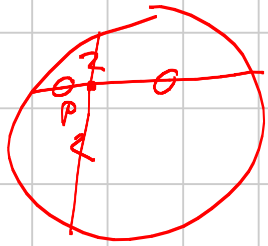
$$\boxed{\overline{QR} = \frac{1}{2}\,\overline{ST}}$$

$\overline{OP} = ?$

Th DELLA CORDA

$$(R + \overset{x}{\overline{OP}}) \cdot (R - \overline{OP}) = \overline{PS}^2$$

$$\boxed{(R+x)\,(R-x) = \overline{PS}^2}$$



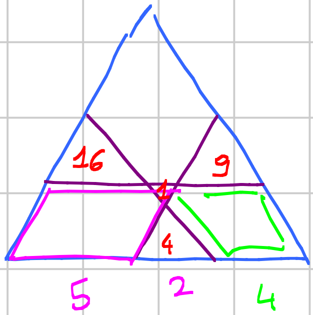
$$\overline{OP} \cdot (R - \overline{OP}) = \overline{PR}^2$$

$R^2 - x^2 = \overline{PS}^2$

$x\,(R - x) = \overline{PR}^2 = \left(\frac{\overline{PS}}{2}\right)^2$

$\overline{PS} = 2\,\overline{PR}$

7.



Aree

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$A(\triangle) = ?$

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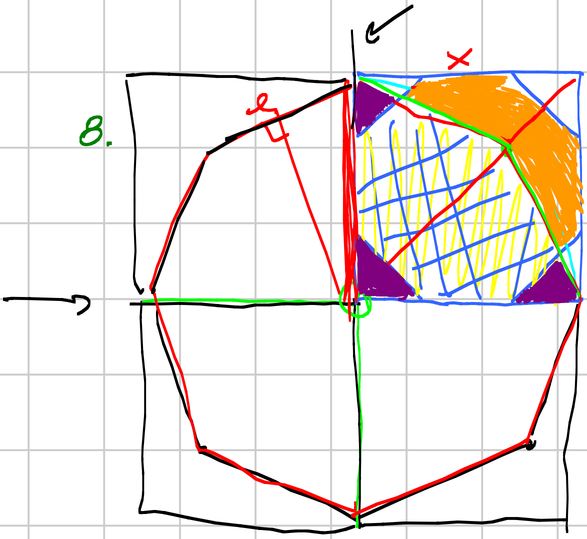
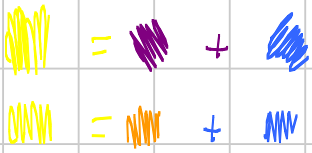
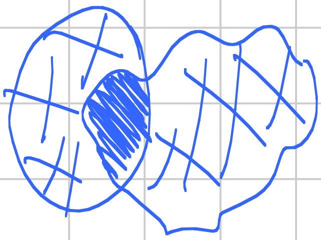
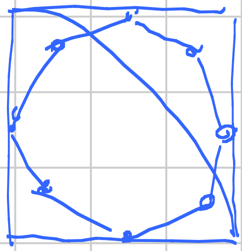


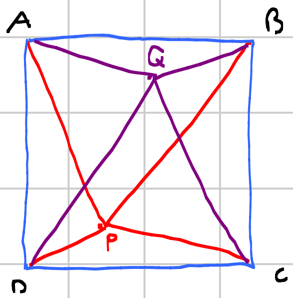
$A - A = ?$







9. PROBLEMA 16 GAS DI IERI



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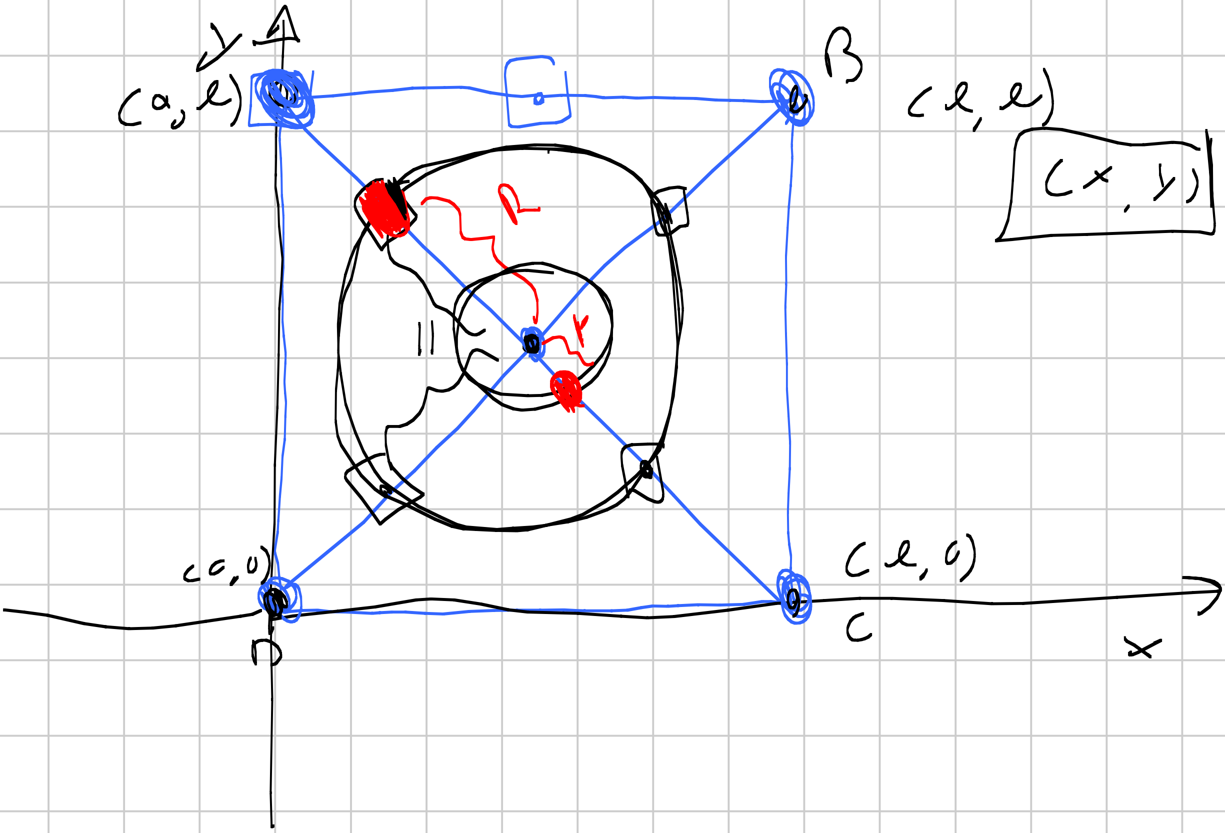

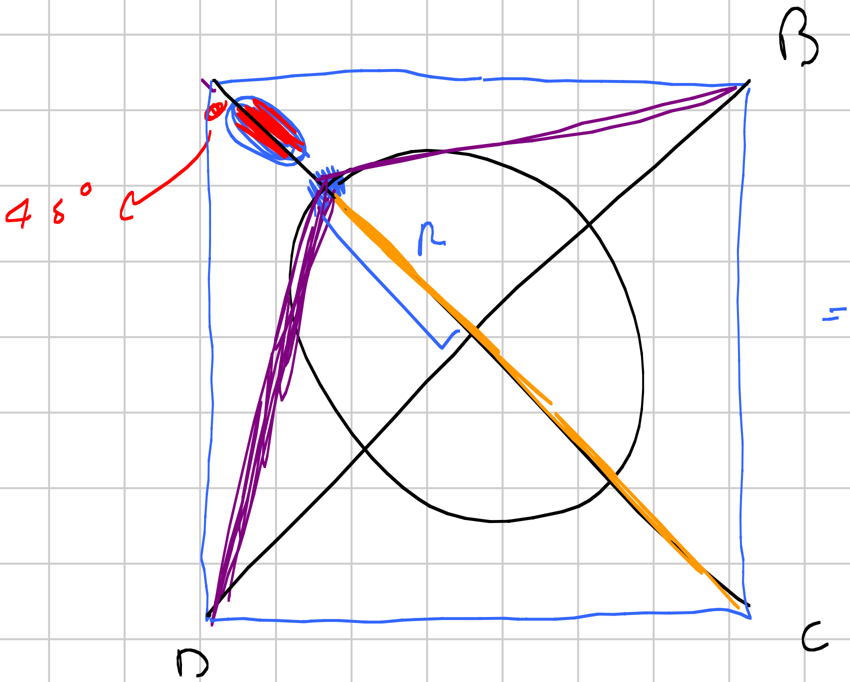

$d_1^2 + d_2^2 + d_3^2 + d_4^2 =$

$= \blacksquare^2 + \blacksquare^2 + \blacksquare^2 + \blacksquare^2$

$r^2 + \_\, r + \_\ \_ = \_\_\_$

Teoria dei numeri / algebra - base -

$\mathbb{N}$ naturali

$\{0, 1, 2, \dots\}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

polinomi

$\mathbb{Z}$ $a, b \in \mathbb{Z}$ $b \neq 0$

Allora $\exists\, q, r \in \mathbb{Z}$ $0 \leq r < |b|$ tali che

$$\boxed{a = qb + r}$$

DIVISIONE EUCLIDEA.

<u>DEF</u> $b \mid a$ (b divide a ; a è diviso da b)

vuol dire $r = 0$ $[\exists q$ t.c. $a = qb]$

<u>OSS</u> Posso cercare di capire che divisori ha un certo numero.

LIMITIAMOCI a $a, b \in \mathbb{N}$

a	
0	$\forall b \in \mathbb{N}$ $b \mid 0$.
1	1
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6

<u>DEF</u> Il $p \in \mathbb{N}$ si dice primo se ha esattamente 2 divisori.

TEO Se p è primo e $p | a \cdot b$ allora $p | a$ oppure $p | b$.

TEO Se $a \in \mathbb{N} \setminus \{0\}$ (cioè a numero naturale, $a \neq 0$)
allora $\exists\, p_1, \dots, p_N$ primi e $\alpha_1, \dots, \alpha_N \in \mathbb{N} \setminus \{0\}$ t.c.

$$a = p_1^{\alpha_1} \cdots p_N^{\alpha_N}$$

e tale scrittura è unica con $p_1 < p_2 < \dots < p_N$.

m.c.m. MINIMO COMUNE MULTIPLO

M.C.D. MASSIMO COMUN DIVISORE

DEF $a, b \in \mathbb{N}$

m.c.m.(a,b) il più piccolo n t.c. $a | n$, $b | n$

M.C.D.(a,b) il più grande m t.c. $m | a$ e $m | b$.

OSS Se $a = p_1^{\alpha_1} \cdots p_N^{\alpha_N}$ con $\alpha_i, \beta_i \in \mathbb{N}$
$b = p_1^{\beta_1} \cdots p_N^{\beta_N}$

allora M.C.D.$(a,b) = p_1^{\gamma_1} \cdots p_N^{\gamma_N}$ con $\gamma_i = \mathrm{MIN}(\alpha_i, \beta_i)$

m.c.m.$(a,b) = p_1^{\delta_1} \cdots p_N^{\delta_N}$ con $\delta_i = \mathrm{MAX}(\alpha_i, \beta_i)$

OSS. M.C.D.$(a,b) \cdot$ m.c.m.$(a,b) = a \cdot b$

OSS 2 M.C.D.$(a,b) | a$
M.C.D.$(a,b) | b$ $\Rightarrow$ M.C.D.$(a,b) | a - b$

Algoritmo di Euclide

M.C.D. $(720, 252)$

$$720 = 2 \cdot 252 + 216$$
$$252 = 1 \cdot 216 + 36$$
$$216 = 6 \cdot 36 + 0$$

$$MCD(720, 252) = MCD(36, 0) = 36$$

Congruenze

Siano $a, b, n \in \mathbb{N}$ $(\mathbb{Z})$ $\boxed{n \neq 0, 1, -1}$

Diciamo che $a \equiv b \ (n)$ [a congruo a b modulo n]

se a e b divisi per n danno lo stesso resto.

$$\begin{cases} a \equiv b & (n) \\ c \equiv d & (n) \end{cases} \Rightarrow \begin{cases} a + c \equiv b + d \\ a - c \equiv b - d \\ a \cdot c \equiv b\,d \end{cases} (n)$$

$$\cancel{\frac{a}{c} \equiv \frac{b}{d}}$$

Riempire la tabella

a	0	1	2	3	4	5	6	7	8	9	10
$2a$	0	2	4	6	1	3	5	0	2	4	6
$3a$	0	3	6	2	5	1	4	0	3	6	2
$-3a+1 \equiv 4a+1$	1	5 / -2	2 / -5	6 / -1	-4	0	-3	1	-2	-5	-1
a^2	0	1	4	2	2	4	1	0	1	4	2
a^3	0	1	1	6 (-1)	1	6	6	0	1	1	6
a^3+1	1	2	2	0	2	0	0	1	2	2	0
2^a	1	2	4	1	2	4	1	2	4	1	2
3^a	1	3	2	6 (-1)	4 (-3)	5 (-2)	1 (-6)	3	2	6	4
3^a+1	2	4	3	0	5	6	2	4	3	0	5
$3a \equiv 2005a$	0	3	6	2	5	1	4	0	3	6	2

$\to 2 \cdot 10 = 2 \cdot 7 + \boxed{6}$ resto!

$\boxed{\text{mod } 7}$

Ultima cifra di 7^{2017} ? $\pmod{10}$

7^{2017}

$7^{2016} \quad 7^8 \quad 7^4 \quad 7^0 \to 1$

$7 \to 9 \to 3 \to 1 \to 7$

(4)

es 1 - 12

---

2 Quanti sono gli interi $1 \le n \le 10'000$ t.c. hanno un numero pari di divisori?

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \qquad p_1 < \dots < p_k \text{ primi}$$

$$d \mid n \iff d = p_1^{\beta_1} \cdots p_k^{\beta_k} \quad \text{con } 0 \le \beta_1 \le \alpha_1, \ \dots, \ 0 \le \beta_k \le \alpha_k$$

scelte totali $(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_k+1)$ = dispari

$\Rightarrow \alpha_1, \dots, \alpha_k$ pari

$\Rightarrow n$ quadrato perfetto

Oppure...

$d \mid n \Rightarrow n = k \cdot d \Rightarrow k \mid n$ $\qquad k = \frac{n}{d}$

Allora $d \mid n \iff \frac{n}{d} \mid n$

Se $d < \sqrt{n} \iff \frac{n}{d} > \sqrt{n}$

se $d = \sqrt{n}$ allora $\frac{n}{d} = \sqrt{n}$

3 Esistono numeri interi con 11 divisori? sì $\quad p^{10}$

$\hookrightarrow 10+1 = 11$ div.

" multipli di 11 con 11 divisori? sì 11^{10}.

Esistono multipli di 22 con 22 divisori?

22 divisori $\qquad 22 = \alpha_1 + 1 \longrightarrow p^{21}$

$22 = (\alpha_1+1)(\alpha_2+1) \longrightarrow p\, q^{10}$

$22 \mid p q^{10} \rightarrow$ $p=2$, $q=11$ o $p=11$, $q=2$

$\boxed{2 \cdot 11^{10}}$ o $\boxed{11 \cdot 2^{10}}$

Esistono multipli di 44 con 44 divisori?

$$\begin{aligned} 44 &= 44 & & \cancel{p^{43}} \\ &= 2 \cdot 22 & & p\, q^{21} \qquad 11 \cdot 2^{21} \\ &= 4 \cdot 11 & & p^3 q^{10} \qquad 2^3 \cdot 11^{10},\ 11^3 \cdot 2^{10} \\ &= 2 \cdot 2 \cdot 11 & & p \cdot q \cdot h^{10} \qquad p \cdot 11 \cdot 2^{10} \end{aligned}$$

$2^2 \cdot 11 \mid n$

3. $\cancel{\dfrac{5n+93}{n+7}} \in \mathbb{N}$ con $n \in \mathbb{N}$

$$\frac{5n+35}{n+7} \in \mathbb{N} \ ?$$

$$\frac{5n+35}{n+7} = \frac{5(n+7)}{n+7} = 5$$

$$\frac{5n+93}{n+7} = \frac{5n+35-35+93}{n+7} = \underbrace{\frac{5n+35}{n+7}}_{=5} + \frac{58}{n+7} \in \mathbb{N}$$

intero se

$n+7 = 1, 2, 29, 58 \Rightarrow n = 22, 51$

$A(x) =$ polinomio

$B(x) =$ polinomio allora posso dividere:

$\exists\, Q(x), R(x)$ (unici) polinomi tali che

$$A(x) = Q(x)\,B(x) + R(x) \quad \text{con grado } R(x) < \text{grado } B(x)$$

$$\text{MCD}\left(x^2, x(x+1)\right) \overset{?}{=} x \qquad = \frac{1}{2}x$$

$x^2 = x \cdot x$

$x(x+1) = x \cdot (x+1)$

$x^2 = \frac{1}{2}x \cdot 2x$

$x(x+1) = \frac{1}{2}x \cdot 2(x+1)$

8 Quanto vale la somma dei divisori positivi di 6^5?

$2^5 \cdot 3^5$ divisori $6 \cdot 6 = \boxed{36}$

Quanto vale la somma dei divisori positivi di 2^5? $2^6 - 1$

$$\underbrace{\underbrace{\underbrace{1+1}_{2}+2}_{2^2}+2^2}_{2^3} + 2^3 + 2^4 + 2^5 = 2^6$$

Quanto fa la somma dei div. positivi di 3^6?

$$1 + 3 + 3^2 + \dots + 3^5$$

Come si fattorizza $(x^{n+1} - 1)$?

$$(x^{n+1} - 1) = (x-1)(x^n + x^{n-1} + \dots + 1)$$

$$1 + x + \dots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

$n = 5$
$x = 3$

$$1 + \dots + 3^5 = \frac{3^6 - 1}{2}$$

E per 6^5 ?

$3 \backslash 2$	1	2	2^2	$\dots$	2^5	
1	1	2	2^2			$1+2+2^2+\dots+2^5 = 2^6-1$
3	3	$2\cdot 3$	$2^2\cdot 3$			$3\,(2^6-1)$
3^2			$\ddots$			$\vdots$
$\vdots$						
3^5						$3^5\,(2^6-1)$

36 divisori

Sommo $\dfrac{(3^6-1)}{2}\,(2^6-1)$

10 Quante sono le terne (x, y, z) t.c. $x, y, z \in \mathbb{N}$

e $x^3 + 5y^3 + 25z^3 = 5xyz$?

$\Rightarrow$ $x^3 \equiv 0 \quad (5)$

$\Rightarrow 5 \mid x^3 \Rightarrow 5 \mid x$ ovvero $x \equiv 0 \ (5)$

$x = 5\ell \quad \ell \in \mathbb{N}$

$(5\ell)^3 + 5y^3 + 25z^3 = 5(5\ell)yz$

$125\ell^3 + 5y^3 + 25z^3 = 25\ell yz$

$25\ell^3 + y^3 + 5z^3 = 5\ell yz$

Continuando $5 \mid x, y, z$

$5^2 \mid x, y, z \quad \ldots \quad 5^n \mid x, y, z \quad \forall n \in \mathbb{N}$

$\Rightarrow$ x, y, z hanno infiniti divisori $\Rightarrow x = y = z = 0$.